

The description of an arbitrary configuration evolution in the Conway's Game Life in terms of the elementary configurations evolution in the first generation

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Abstract

The connection between the evolution of an arbitrary configuration and the evolution of its parts in the first generation is established. The equivalence of Conway's evolution rules to the elementary configurations' (containing one, two, three, and four pieces) evolution laws in the first generation has been proved.

1 Introduction

Let us consider, for example, a two-dimensional configuration of pieces shown in Fig. 1a. Figure 1b sets up a correspondence between the pieces and the letters by which we denote each piece. Two cells in the plane are neighboring ones if they have a common side or vertex. A piece in the cell C , for instance, has three adjacent pieces.

Conway's rules for a configuration evolution in the first generation are [1]:

1. **Survival:** if a piece has two or three adjacent pieces, it transfers to the next generation.
2. **Death:** otherwise a piece dies.
3. **Birth:** if an empty cell has exactly three neighboring pieces, a piece is created in this cell in the next generation.

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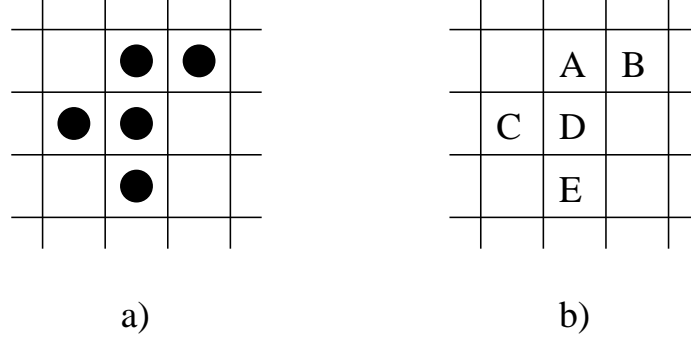


Figure 1: Initial configuration.

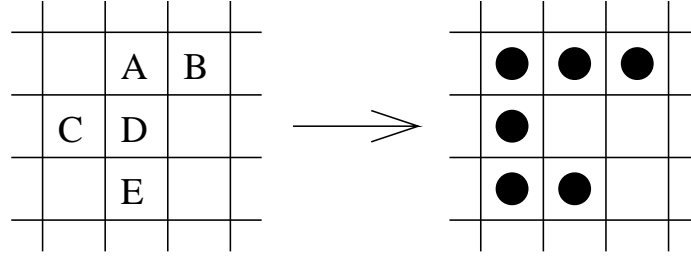


Figure 2: First evolution step for the initial configuration.

Rules 1, 2, and 3 are applied simultaneously and describe one "step". For instance, Figure 2 illustrates one step (first generation) in the evolution of the Fig. 1 configuration.

2 Definition of The Basic Concepts

Let us discuss, for example, evolution of the configurations a to e shown in Fig. 3, which are defined as 1-*fragments* and can be obtained by removing one of the pieces from the original configuration (Fig. 1). When the piece A is removed from the original configuration, we get the initial fragment given by Fig. 3a, and so on.

Now let all the patterns evolve in the first generation and count the number of pieces in each cell. These numbers are given in Fig. 4.

The number of pieces in cell F is denoted by $S(F)$. One can see that the first generation of the Fig. 1 configuration shown in Fig. 2 can be obtained by putting pieces to the cells with $S = 2$ and 5, and leaving empty the

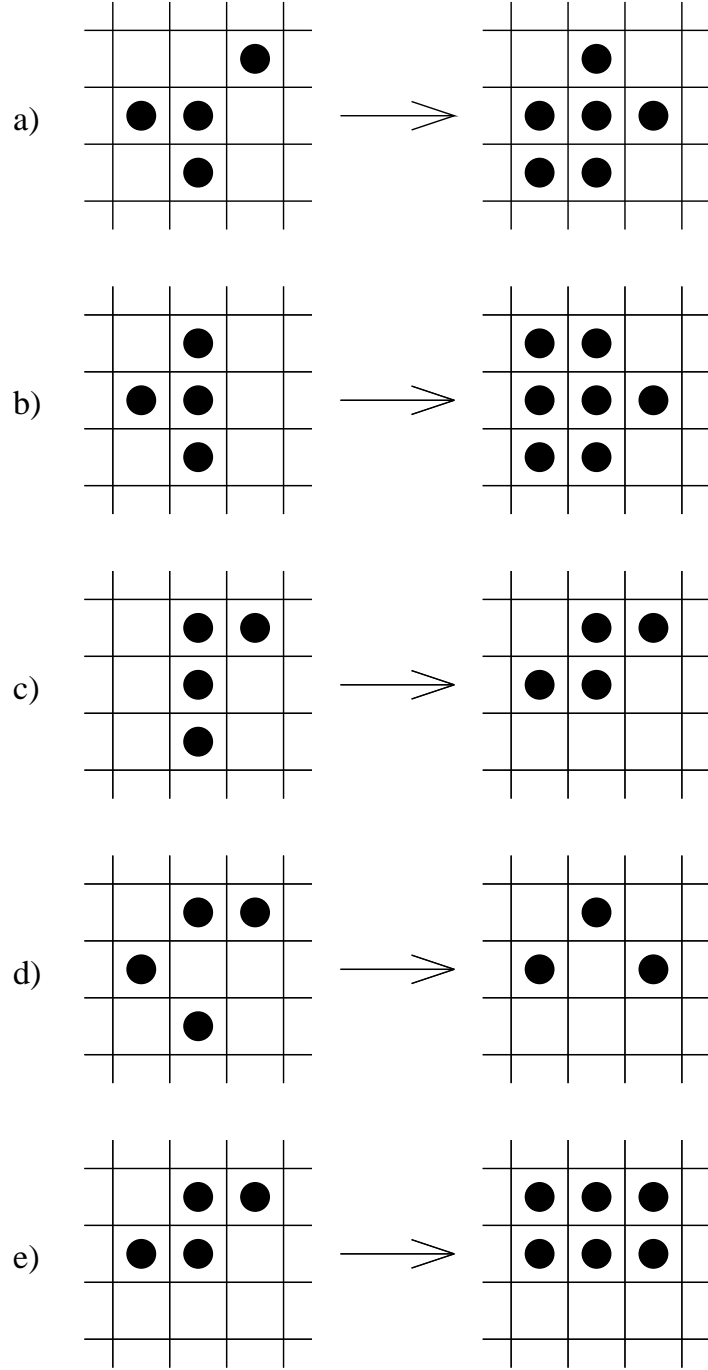


Figure 3: First evolution step of all 1-fragments of the initial configuration.

2	5	2	
5	4	4	
2	2	0	

Figure 4: Values of $S(F)$ for 1-fragments in the first generation.

cells corresponding to $S = 0$ and 4. Thereby we have reduced the evolution of the configuration containing five pieces to the evolution of the patterns containing four pieces (Fig. 3). We call this procedure $5 \rightarrow 4$ *reduction*.

3 $N \rightarrow N - 1$ Reduction Theorem (Removal of One Piece)

Suppose that the initial pattern is composed of N pieces. Let us consider the first generation evolution of its 1-fragments which contain $N - 1$ pieces each and establish the correspondence between the presence or absence of a piece in the cell in the first generation and the value of $S(F)$ in this cell.

1. Survival:

- (a) if a piece F has exactly two pieces in its neighboring cells then this cell is filled in the first generations of all 1-fragments except for the three fragments obtained by removing from the initial pattern either the piece F or one of its neighbors, i.e., $S(F) = N - 3$;
- (b) if a piece F has three pieces in its adjacent cells then its cell is filled in the first generations of all 1-fragments, i.e., $S(F) = N$.

2. Death:

- (a) for any number of pieces in the cells adjacent to a piece F except 4 (i.e., at 0, 1, 5, 6, 7, and 8), its cell is empty in the first generations of all 1-fragments, hence $S(F) = 0$;
- (b) if a piece F has four neighbors then it is present in the first generation of four 1-fragments obtained by removing from the initial pattern one of the four pieces adjacent to F , thus in this case $S(F) = 4$.

3. Creation:

- (a) if an empty cell F in the initial configuration has three neighboring pieces then a new piece is generated in this cell in the first evolution step of all 1–fragments, except for the three patterns obtained by removing one of these three pieces, i.e., $S(F) = N - 3$;
- (b) if an empty cell F has four pieces in its adjacent cells then $F = 4$;
- (c) at any other number of pieces in the cells adjacent to an empty cell, a piece cannot be generated in this cell in the first generation of all 1–fragments, hence $S(F) = 0$.

These results are summarized in the convenient form in Table 1. This table clearly shows that the numbers $S(F) = N - 3$ and N correspond to the presence of a piece in the cell, whereas $S(F) = 0$ and 4 corresponds to the absence of a piece.

Evolution process	The # of neighboring cell F pieces at which the process takes place	$S(F)$ corresponding to 1– fragments	$S(F)$ corresponding to 2– fragments
1. Survival	2	$N - 3$	$(N - 3)(N - 4)/2$
	3	N	$(N + 3)(N - 4)/2$
2. Death	0, 1, 6, 7, 8	0	0
	4	4	$4(N - 4) + 6$
	5	0	10
3. Birth	3	$N - 3$	$(N - 3)(N - 4)/2$
4. Absence of birth	0, 1, 2, 6, 7, 8	0	0
	4	4	$4(N - 4)$
	5	0	10

Table 1. The rules of correspondence between $S(F)$ and the presence or absence of a piece in the cell F for $N \rightarrow N - 1$ and $N \rightarrow N - 2$ reductions.

At the same time, if $N - 3 = 0$, or $N - 3 = 4$, or $N = 4$, i.e. $N = 3, 4$, or 7, we cannot establish one-to-one correspondence between the values of $S(F)$ and the presence or absence of a piece in the cell F in the first generation. Configurations containing these numbers of pieces we name 1–*irreducible* (i.e., they cannot be analyzed by removing one piece).

Thus, the evolution of an arbitrarily complex configuration of more than seven pieces can be analyzed on the base of the first generation evolution rules for various configurations with seven pieces and the correspondence rules summarized in the Table 1.

Now we can formulate the $N \rightarrow N - 1$ reduction theorem. *Conway's evolution rules are equivalent to the combination of the rules for one evolution step of 1–irreducible configurations (i.e., all configurations composed of*

$N = 3, 4$, and 7 pieces - whereas the fact that configurations with $N = 1$ and 2 annihilate after the first step is taken for granted) and the correspondence rules formulated as follows: if $S(F) = N - 3$ or N , a piece is generated (conserved) in the cell F , otherwise cell F must be empty after the first step.

4 $N \rightarrow N - M$ Reduction Theorem (Elimination of M Pieces)

Alongside with the original configuration composed of N pieces, let us consider one evolution step of C_N^M (number of the combinations where one can choose M of N) M -fragments containing $N - M$ pieces each.

The rules of correspondence for the general case of M -fragments are summarized in Table 2. These rules can be epitomized in the following statement: if $S(F) = C_{N-3}^M$ or C_{N-4}^{M-1} then there is a piece in the cell F after one evolution step, otherwise the cell is vacant (it is obvious that the rule applies only to M -reducible configurations). It is assumed in Table 2 that $C_I^K = 0$ if $I < 0$ and/or $K < 0$.

Evolution process	The # of neighb. F pieces at which the process occurs	$S(F)$ corresponding to M -”fragments”	$S(F)$ corresponding to $M = N - 4$
1. Survival	2 3	$C_3^0 C_{N-3}^M$ $C_4^0 C_{N-4}^M + C_4^1 C_{N-4}^{M-1}$	$N - 3$ $1 + 4(N - 4)$
2. Death	0, 1 $L = 4, 5, \dots, 8$	0 $C_L^{L-3} C_{N-L}^{M-(L-3)} +$ $C_L^{L-2} C_{N-L-1}^{M-(L-2)}$	0 $L(L - 1)/6$ $[(L + 1)(N - L) - 3]$
3. Birth	3	$C_3^0 C_{N-3}^M$	$N - 3$
4. Absence of birth	0, 1, 2 $L = 4, 5, \dots, 8$	0 $C_L^{L-3} C_{N-L}^{M-(L-3)}$	0 $L(L - 1)(L - 2)$ $(N - L)/6$

Table 2. The same as in Table 1 for the general case of $N \rightarrow N - M$ reduction and $N \rightarrow 4$ reduction.

Examples of the reasonings according to which Table 2 has been composed are given below.

- A) **Survival.** Suppose that a piece in the cell F has three neighbors. Let us select a subconfiguration of the initial configuration that includes a piece in the cell F and its neighboring pieces. This piece survives the next evolution step in two cases only. Firstly, when we remove only pieces not included in the subconfiguration then the resulting number of M -fragments is $C_4^0 C_{N-4}^M$. Secondly, when we remove only one

piece of the subconfiguration then the number of these M -fragments is $C_4^1 C_{N-4}^{M-1}$.

B) **Death.** Suppose that a piece in the cell F has four neighbors. In this case, we include into subconfiguration only the pieces adjacent to F . A piece F will remain only in the following cases:

- (1) One of the pieces in the subconfiguration is removed. The number of such M -fragments is $C_4^1 C_{N-4}^{M-1}$.
- (2) Two pieces included in the configuration are removed and a piece F is saved. In this case the number of M -fragments is $C_4^2 C_{N-5}^{M-2}$.

The correspondence rules for 2-fragments are given in Table 1. By equating, as previously, the values of $S(F)$ corresponding to the presence and the absence of a piece at the cell F , we come to the conclusion that the configurations of $N = 3, 4, 5, 8$, and 11 pieces are 2-irreducible (they cannot be analyzed by removing two pieces). An important point, however, is that the configurations containing seven pieces are 2-reducible.

Thus, despite the fact that the configurations with $N = 7$ are 1-irreducible they turn out to be 2-reducible (a piece occupies the cell F if, and only if, $S(F) = 6$ or 15). It means that we do not have to know the evolution laws for the configurations of seven pieces which were necessary according to the $N \rightarrow N - 1$ reduction theorem.

Since all configurations with $N > 4$ are $(N - 4)$ -reducible (as follows from Table 2), $N \rightarrow 4$ reduction theorem has, probably, the most rational form. The correspondence rules for this case are listed in Table 2.

$N - 4$ reduction theorem: *Conway's three evolution rules are equivalent to the postulated evolution laws of all configurations containing $N = 3$ and 4 pieces in the first generation (configurations with $N = 1, 2$ annihilate after the first step) and the rule of correspondence: a piece is generated or conserved at the cell F if, and only if, $S(F) = N - 3$ or $1 + 4(N - 4)$.*

This theorem prescribes how to reduce the evolution of an arbitrarily complex configuration to the evolution of the elementary configurations containing $N = 4$ in the first generation.

5 Discussion of The Results

Conway's rules are expressed in the local form. Indeed, the presence or absence of a piece, say, in the cell D in the first generation of the configuration $A...E$ (Fig. 2) is determined exclusively by the state of the adjacent cells. On the contrary, the rules based in the discussed theorem have the global

form because they are based on the laws of evolution of the configurations containing four pieces separated by an arbitrary distance.

Physically, one can say that the fundamental "interaction" in the Conway's game Life is the four-piece "interaction". Here we consider three-piece "interaction" as a particular case of the four-piece "interaction" when one of the pieces is well separated from the other three (we also assume that one- and two-piece "interactions" vanish in the first generation).

Finally, we note that our theorems have been formulated strictly in accordance with the Conway's evolution rules. It is possible to formulate similar global rules for any cellular automata.

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References

- [1] M. Gardner, *The Unexpected Hanging and Other Mathematical Diversions*, Simon and Schuster, New York (1969).